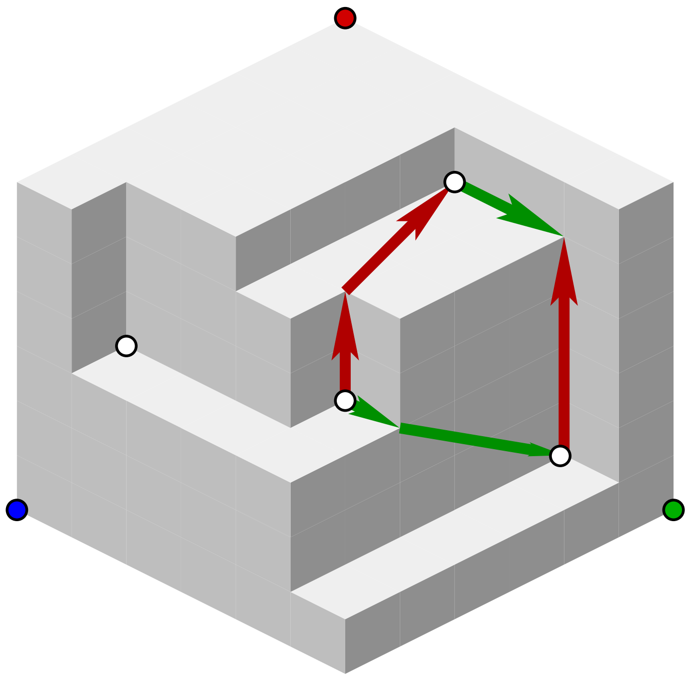
Order & Geometry Workshop

Gułtowy Palace, September 13 – 17, 2016

Problem booklet



7. Oktober 2016

Preface

The workshop was a gathering of 35 people working in combinatorics and theoretical computer science with special emphasis on discrete geometry, partially ordered sets and all kinds of interplays between them.

The workshop was a problem oriented. This booklet contains the problems presented by participants together with some notes on progress. Further problems can be found in the slides of evening talks on the webpage. During the workshop we had the special delight of celebrating the 60th birthday of Jerry Spinrad.

The workshop took place in Gułtowy Palace near Poznań in Poland as a satellite event to the 6th Polish Combinatorial Conference, September 19-23, 2016, in Będlewo.

Organizers

- Stefan Felsner, Technische Universität Berlin
- Piotr Micek, Jagiellonian University, Kraków

Web page

- `http://orderandgeometry2016.tcs.uj.edu.pl`

Former events

- Order & Geometry Workshop, Berlin, August 12-15, 2013
- Problems in Combinatorics and Posets session, Kraków, September 12-15, 2012

Problems

Michał Lasoń

Let M be a matroid of rank r on a ground set E of size $kr+1$, for $k \geq 2$. Let $x, y \in E$ be two elements such that both sets $E \setminus x$ and $E \setminus y$ can be partitioned into k disjoint bases. Do there exist such partitions of these sets which share a common basis?

We conjecture that the answer is 'yes' (Conjecture 13, arXiv:1601.08199v2), and prove it in the case $k \geq 2^{r-1}+1$ (Proposition 15, arXiv:1601.08199v2).

Louis Esperet

The adjacency poset of a graph $G = (V, E)$, introduced by Felsner and Trotter, is the poset $(P_G, \leq)$ with ground set $V \cup V'$ (where V' is a copy of V), in which $u \leq v$ if $u = v$ or if the vertices corresponding to $u \in V$ and $v \in V'$ are adjacent in G.

The boxicity of a graph G is the minimum d such that G can be represented as the intersection of d-boxes in $\mathbb{R}^d$ (a d-box is the cartesian product of d intervals of $\mathbb{R}$).

The boxicity of planar graphs and graphs embedded in fixed surfaces is quite well understood. So is the boxicity of graphs of bounded treewidth. So the next class of interest (by inclusion) is the class of graphs excluding a fixed minor.

Question: what is the boxicity of a graph with no K_t-minor?

Since we know that the dimension of the adjacency poset is at the most twice the boxicity + the chromatic number + 4, any reasonable bound for the boxicity would translate into a bound for the dimension of the adjacency poset.

Using a connection with the acyclic chromatic number (discovered by Felsner and Trotter) and results of Nesetril and Ossona de Mendez, it is not difficult to obtain $O(t^4(logt)^2)$. Can we do better? A random graph construction shows that one cannot expect better than $t\sqrt{\log t}$, so the gap is quite large.

Progress

We are able to improve the upper bound to $O(t^3)$. As a tool we use *r-weak coloring numbers*, which can be seen as a generalization of the degeneracy of a graph. For example, if G is a graph and its 2-weak coloring number $\mathrm{wcol}_2(G)$ is bounded by c, then this means that there is a linear order pi on the vertices of G such that for all $v \in V(G)$, the number of vertices that are preceding v in π and that are reachable from v by a path of length at most 2 is less than c.

We can show the following relation.

Theorem. *For all graphs G, we have* $\mathrm{box}(G) \leq 2\,\mathrm{wcol}_2(G)$.

By a result of van den Heuvel et al. we know that the 2-weak coloring number of K_t-minor free graphs is at most $O(t^3)$. Together with the previous theorem, this implies the claimed progress on the problem.

Theorem. *For all K_t-minor free graphs G, we have* $\mathrm{box}(G) = O(t^3)$.

Stefan Felsner

I have collected some problems about dimension where the complexity status is unknown.

- Dimension of interval orders.
  (Small interval orders of dimension 4).
- Dimension of orders of width 4.
- Dimension of an oder of type $P + C$ where P is 2-dimensional and C is a chain.
- Dimension of graphs (either in the version of graph dimension or as dimension of the incidence order).
- Dimension of grid intersection graphs (we know they are at most 4-dimensional).

Here is a more geometric one:

Does every bipartite 3-dimensional order admit a separated representation?

A separated representation is an order preserving embedding into $\mathbb{R}^3$ such that the set of minima and the set of maxima of the order can be separated by a plane with normal $\mathbb{1}$. This relates to triangle containment vs. triangle intersection.

A Problem of Sergio Cabello

- Communicated by Michał Lasoń.
- Scribe: Stefan Felsner

The paper *Refining the Hierarchies of Classes of Geometric Intersection Graphs* by Sergio Cabello and Miha Jejčič (`arXiv:1603.08974v1`) concludes with the following problem: Is the class of outer-segment graphs a strict subclass of ray graphs? The first author conjectures that this is the case.

The relevant definitions are:

outer-segment: A graph is an outer-segment graph if it has an intersection model consisting of straight-line segments lying in a disk such that each curve has one endpoint on the boundary of the disk.

ray: A graph is a ray intersection graph if it is the intersection graph of rays, or equivalently halflines, in the plane.

The conclusion continues with the following related question: Consider the class $\mathscr{A}$ of intersection graphs of downward rays, that is, halflines that are contained in the halfplane below the x-axis. Let $\mathscr{B}$ be the class of grounded segments, that is, segments contained in the halfplane below the x-axis with one endpoint on the x-axis. It is easy to see that $\mathscr{A}$ is a subset of $\mathscr{B}$ Is the containment proper?

Progress

The following results where obtained during the workshop.

- $\mathscr{A} = \mathscr{B}$, i.e., the classes of downward ray graphs and of grounded segment graphs coincide.
- There are outer-segment graphs which have no representation with rays.
- There are ray graphs which have no representation with downward rays.

Tillmann Miltzow

Subset Token Swapping on a Path and Bipartite Minimum Crossing Matchings

Sometimes it happens that some seemingly completely unrelated problems are in fact equivalent. We first define them and then explain why they are equivalent.

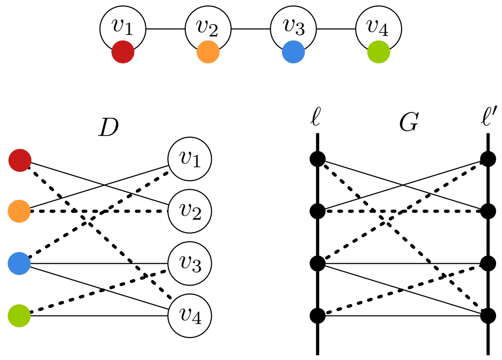
Subset Token Swapping
Input:

1. an undirected connected graph G with n vertices $V = \{v_1, \ldots, v_n\}$.
2. a set of tokens $T = \{t_1, \ldots, t_n\}$
3. initially token t_i is placed on vertex v_i.
4. a function $D : T \to 2^V$, specifying for token t_i the destinations $D(t_i) \subset V$.
5. a number $k \in \mathbb{N}$.

Question: Does there exists a sequence of at most k swaps such that after performing every token is at one of its destination vertices? Here a swap, is an exchange of two tokens on a pair of adjacent vertices.

Bipartite Minimum Matching
Input:

1. two parallel lines ℓ and ℓ'
2. a plane bipartite graph G with all black vertices on ℓ and all white vertices on ℓ'.
3. a number $k \in \mathbb{N}$.



The number of required swaps equals the number of inverions of the target permutation π, which in turn equals the number of crossings of the corresponding plane graph.

Question: Does there a perfect matching $\mathcal{M} \subseteq G$ with at most k crossings?

To see that these two problems are equivalent note that a valid target configuration for the tokens corresponds to a perfect matching in the vertex-token graph or equivalently a permutation π, see Figure 0.1. If the underlying graph is a path, we know that the total number of swaps to reach π equals the number of steps in bubble sort to sort π. It is well-known that the length of the sequence equals the number of inversions of π, which in turn is equivalent to the number of crossings in case that π is interpreted as a perfect matching. We ask if this algorithmic problem is NP-hard.

Warning: We studied the token swapping problem already intensively. If you are interested in the general problem it might be beneficial to consult me first, to see what is open and what is already known.

Jerry Spinrad

I have a number of questions I am interested in.

The general structure is based on the following generalization of transitivity, which I called triangle–extendability in my book on efficient graph representations.

A characterization of a transitively orientable graph is that the vertices can be ordered $x_1, \ldots, x_n$ such that whenever you select 3 vertices, the only non-edge cannot be from the first to the last vertex in the ordering.

In triangle–extendability, this definition is changed so that whenever you select 4 vertices, you cannot have the only non-edge being from the 1st to the last in the ordering.

The classes I am interested in at the moment (though many other classes could be considered; for example chordal and co-triangle extendible instead of chordal and co-comparability) involve generalizations of permutation graphs/2 dimensional posets.

There are 2 natural ways to generalize permutation graphs.

1) The class such that both G and G complement are triangle–extendible
2) Require a single ordering of the vertices which is a triangle–extendible ordering for

both G and G complement

These 2 classes are equivalent (both characterizations of permutation graphs) when we use transtitively orientable, but are different when we use triangle–extendability.

Most of the natural questions are open for both classes, though given the ordering you can solve clique and independent set in polynomial time. I am particularly interested in these questions

Recognition.
Counting (the number of graphs in the class on n vertices is $2^{\Theta f(n)}$ for which $f(n)$?
Characterizations and representations of the class.

A perhaps more tractable question I have come up with recently is whether all graphs in the class have either a clique or indepenent set of size $cn^{1/2}$ as permutation graphs do; this would put an upper bound on the number of graphs in the class which is somewhat better than any obvious upper bound I see. Of course, of the 2 classes, the $2d$ class is smaller and probably easier to work with.

Veit Wiechert and Gwenaël Joret

The dimension of dense bipartite graphs

Given a bipartite graph $G = (A \cup B, E)$, we can associate with G a poset P_G of height 2. Namely, we view A as the minimal elements, B as the maximal elements, and we have that $a \in A$ is below $b \in B$ in P_G iff a and b are adjacent in G. Now it is natural to define the dimension of G as the (poset) dimension of P_G.

Our goal is to establish a connection between the edge-density of bipartite graphs and their dimension. Clearly, in the current setting we get a problem as complete bipartite graphs have large density but small dimension. However, if we remove a matching from these graphs, then we obtain large standard examples, which have large dimension. So to overcome this issue, we do not only consider the dimesion of the bipartite graph itself, but also the dimension of its subgraphs. Thus, given a bipartite graph G we let

$$\dim_s(G) := \max_{H \subseteq G} \dim(H).$$

We are interested in the following question: Do bipartite graphs with large density have large $\dim_s$? More precisely, if $\mathcal{G}$ is a class of bipartite graphs with unbounded average degree, is it true that then $\dim_s$ must be unbounded on graphs of $\mathcal{G}$?

If this is indeed the case, then this would solve the conjecture that graph classes with bounded expansion can be characterized in terms of poset dimension.

Tomasz Łuczak

Conjecture Let P be a poset of height 2, with dimension at least $d \geq 13$ and girth at least d^d. Then for every $P' \subseteq P$ we have $\dim P' \leq \dim P$.

Casey Tompkins

Let $La(n, P)$ be size of the largest collection of subsets of $[n]$ not containing P as a weak subposet with respect to the containment relation. Let $e(P)$ be the maximum number k, such that for all n the k middle levels of $2^{[n]}$ do not contain P as a subposet. It is conjectured that

$$\lim_{n\to\infty} \frac{La(n, P)}{\binom{n}{n/2}} = e(P).$$

A reasonable step in this direction is to prove that there is some function f depending only on $e(P)$ so that

$$La(n, P) \leq f(e(P))\binom{n}{n/2}.$$

Kolja Knauer

For every *fixed* $k \geq 3$, it is NP-complete to decide if a given poset P has dimension at least k. The set of vertices of the *linear extension graph* $G_L(P)$ of P is the set of all linear extensions of P and there is an edge between L and L' if and only if L and L' differ by a neighboring transposition, i.e., by reversing the order of two consecutive elements.

If instead of the poset its linear extension graph is considered to be the input of the dimension problem, then we get:

Observation 1. *The dimension of a poset P can be computed in quasi-polynomial-time in $n = |G_L(P)|$.*

Beweis. It is a classic result that $\dim(P) \leq \omega(P)$, where $\omega(P)$. Since any permutation of an antichain appears in at least one linear extension, $\omega(P)! \leq n$ and therefore $\dim(P) \leq \log(n)$. Thus, to determine the hull-number of $G_L(P)$ it suffices to compute the intersection poset of all subsets of at most $\log(n)$ linear extensions. Since the intersection poset can be computed in polynomial-time, we get the claimed upper bound. $\square$

The question is, whether quasi-polynomial-time is best-possible. In fact in the paper *Convexity in Partial Cubes: the Hull Number* with Marie Albenque we made the following:

Conjecture 1. *The dimension of a poset given its linear extension graph can be determined in polynomial-time.*

Patrice Ossona de Mendez

- Scribe: Stefan Felsner

Let Δ be an abstract simplicial complex. It is known that Δ has a geometric realization in some dimension. In [POM] it is shown that the order dimension gives an upper bound on the dimension of a realization space. More precisely:

$$\dim(P_\Delta) \leq d \quad \implies \Delta \text{ is realizable in } \mathbb{R}^{d-1}.$$

The proof is based on a construction due to Scarf and shows that Δ has a realization as a subcomplex of a simplex. The equivalence in the case $d = 3$ was shown by Schnyder.

Problem: Find a geometric characterization of simplicial complexes that have no realization in $\mathbb{R}^{d-1}$ for $d < \dim(P_\Delta)$.

[POM] P. Ossona de Mendez, *Realization of posets*, Journal of Graph Algorithms and Applications, 6 (2002), pp. 149–153.

Rado Fulek

Picture Hanging Puzzle

We simplified a little bit a question from [DD+14].

Let S denote a sequence over alphabet $\Sigma = \{a_1, \ldots, a_n\}$. We require that every element of Σ occurs at least once in S. A **simplification** of S is an operation that deletes from S two consecutive occurrences of the same element of Σ. The sequence S is **trivial** if we can obtain an empty sequence from S by successively applying simplifications. A **restriction** of S to a subset $\Sigma' \subset \Sigma$ is a sequence S' obtained from S by deleting every element in $\Sigma \setminus \Sigma'$.

What is the order of magnitude of the function $f(n)$ defined as the minimum value such that there exists S over Σ, $|\Sigma| = n$, of length $f(n)$ with the following property. For every $a_i \in \Sigma$ the restriction of S to $\Sigma \setminus \{a_i\}$ is trivial. It is known that $f(n) \in O(n^2)$ [DD+14] and $f(n) \in \Omega(n2^{\sqrt{\log n}})$ (unpublished).

[DD+14] Erik D Demaine, Martin L Demaine, Yair N Minsky, Joseph SB Mitchell, Ronald L Rivest, and Mihai Patraşcu. *Picture-hanging puzzles.* Theory of Computing Systems, 54(4):531–550, 2014.

Jerry Griggs

Finding Diamonds in the Middle Three Layers

It interesting to study the maximum size of a family of subsets of an n-set that contains no (weak) copy of a given poset P, denote this $La(n,P)$. It is conjectured [GriLu,GriLiLu] that $La(n,P)$ is asymptotic to $\binom{n}{\lfloor n/2 \rfloor}$ times some integer $e(P)$ depending on P, as $n \to \infty$. While this has been proven for several posets P, it remains a tough challenge, even for the four-element diamond poset. When P is the diamond, the conjecture says that $La(n,P) \sim 2\binom{n}{\lfloor n/2 \rfloor}$. Taking all sets of two middle sizes gives a diamond-free family of of the conjectured asymptotic size. The upper bound has been brought down in a series of papers by various authors, with current best bound [GroMetTom2] being $2.20711\binom{n}{\lfloor n/2 \rfloor}$.

Restricting attention to just the middle three levels should make the problem appealing to a broader audience, as now the no-diamond problem is that of bounding the largest size of a family of vertices in the middle three slices of the hypercube that induces no C_4.

For simplicity, let's assume n is even, say $2k$, and we consider families $\mathscr{F}$ of subsets of $[n]$, where the subsets have only the three middle sizes, $k+1, k, k-1$. Let us say the proportions of all subsets of those sizes that we have in $\mathscr{F}$ are denoted by P, Q, R, respectively. The conjecture is that for every $\varepsilon > 0$, for all sufficiently large n, if $P + Q + R > 2 + \varepsilon$, then $\mathscr{F}$ contains a diamond. In this restricted case, we are still rather far from a proof (though most experts continue to believe the conjecture). The best upper bound in this case is currently $2.15121\binom{n}{\lfloor n/2 \rfloor}$ [Bal] (also see [ManShe]), which we seek to improve.

A diamond in the middle three levels consists of two k-subsets at Hamming distance two, along with their union (of size $k+1$) and their intersection (of size $k-1$). Let us note that for any subset X of the lower level and subset Y of the upper level, with $X \subset Y$, there are precisely two subsets in the middle level between them, and these four sets together form a diamond. Define the Johnson graph $J = J(n,k)$ (with $k = n/2$) to have as vertices the $n/2$-subsets of $[n]$, with edges joining any two vertices (subsets) at Hamming distance two. We see that diamonds in the middle three levels correspond to edges in the Johnson graph.

Each subset in the upper level, those of size $k+1$, has a down-shadow in the middle level of $k+1$ subsets (each obtained by deleting an element), and those subsets form a clique in J. Because an edge of J is below a unique upper level subset, it means that the down-shadows of the upper level subsets decompose the edges of J into edge-disjoint cliques, each on $k+1$ vertices. Two of the cliques may share a vertex of J, but never more than one. Dually, the up-shadows of the lower level subsets decompose the edges of J into edge-disjoint cliques, each on $k+1$ vertices. A clique from a down-shadow and a clique from an up-shadow have at most two vertices in common; When they have two, the vertices form an edge in J, and there is a diamond consisting of those two subsets along with the upper level set and the lower level set that generated the cliques.

We presented ideas in a particular case, $P = R = .9$ for obtaining a value of Q to ensure there must be a diamond in family $\mathscr{F}$ (we can use $Q > .4$), based on a worst-case scenario for how these two clique decompositions interact with the k-sets in $\mathscr{F}$. We hope that further insights can lead to improved bounds on Q in this sample case, and then for general P, R.

Ryan Martin and Libby Taylor are pursuing a somewhat different approach, deriving properties of a *maximal* three-layer diamond-free family, working out from the middle layer Johnson graph (instead of from the outer layers in). Their goal is to obtain inequalities involving the degrees of the sets in the family, viewed as graph vertices, to bound the family size.

[AxeManMar] M. Axenovich, J. Manske, and R. Martin, Q_2-free families in the Boolean lattice, *Order* **29**, (2012), 177–191.

[Bal] J. Balogh, P. Hu, B. Lidický, and H. Liu, Upper bounds on the size of 4- and 6-cycle-free subgraphs of the hypercube, *European Journal of Combinatorics* **35** 2014, 75-85.

[GriLi] J. R. Griggs and W.-T. Li, Progress on poset-free families of subsets, in IMA Springer volume, 2016.

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[GriLiLu] J. R. Griggs, W.-T. Li, and L. Lu, Diamond-free families, *J. Combinatorial Theory (Ser. A)* **119** (2012), 310–322.

[GriLu] J. R. Griggs and L. Lu, On families of subsets with a forbidden subposet, *Combin. Probab. Comput.* **18** (2009), 731–748.

[GroMetTom] D. Grósz, A. Methuku, and C. Tompkins, An improvement of the general bound on the largest family of subsets avoiding a subposet P, ArXiv:1408.5783.

[GroMetTom2] D. Grósz, A. Methuku, and C. Tompkins, An upper bound on the size of diamond-free familes of sets, ArXiv:1601.06332.

[Kat1] G. O. H. Katona, Forbidden inclusion patterns in the families of subsets (introducing a method), in *Horizons of Combinatorics*, Bolyai Society Mathematical Studies, **17**, Bolyai Mathematical Society, Budapest and Springer-Verlag, 2008, pp. 119–140.

[KatTar] G. O. H. Katona and T. G. Tarján, Extremal problems with excluded subgraphs in the n-cube, in: M. Borowiecki, J. W. Kennedy, and M. M. Sysło (eds.) **Graph Theory**, Łagów, 1981, *Lecture Notes in Math.*, **1018** 84–93, Springer, Berlin Heidelberg New York Tokyo, 1983.

[KraMarYou] L. Kramer, R. R. Martin, and M. Young, On diamond-free subposets of the Boolean lattice, *J. Combinatorial Theory (Ser. A)* **120** (2013), 545–560.

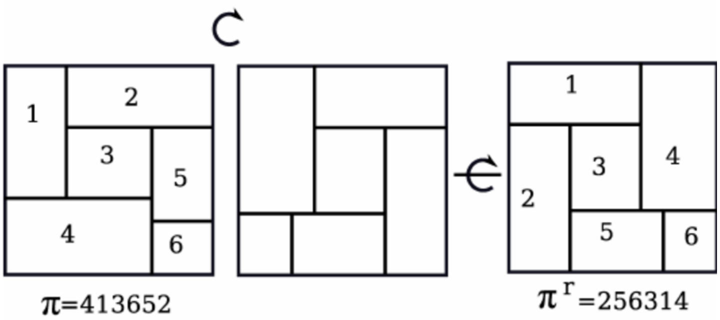
[ManShe] J. Manske and J. Shen, Three layer Q_2-free families in the Boolean lattice, *Order* **30** (2013), 585–592.

Jean Cardinal

Baxter d-permutations

Guillotine partitions are dissections of the unit square into rectangles obtained by recursively splitting by a vertical or horizontal segments. The combinatorial structures of those partitions are in one-to-one correspondence with so-called *separable permutations*. Separable permutations can be defined as those avoiding the patterns 2413 and 3142. Similarly, separable permutations can be used to define the corresponding *series-parallel* orders, the family of two-dimensional partial orders obtained by series and parallel compositions.

This can be generalized to more general dissections of the square. Let us consider the equivalence classes obtained on arbitrary dissections by shifting the segments in a way that preserves their incidence relation. In 2005, Ackerman, Barequet, and Pinter gave an elegant bijection between those equivalence classes and *Baxter permutations* [ABP06]. An example is given in the figure. Baxter permutations can be defined as avoiding some simple vincular patterns.



Examples of floorplans and their corresponding Baxter permutations (from Balachandran and Koroth, 2011).

In 2008, Asinowski and Mansour generalized separable permutations to separable d-permutations, and showed those were in bijection with 2^{d-1}-dimensional guillotine partitions [AM10]. They posed the question of describing a class of d-permutations which would generalize Baxter permutations in the sense of being in a natural bijection with 2^{d-1}-dimensional partitions. In particular, can we generalize the bijection from Ackerman et al. [ABP06] to higher-dimensional partitions?

[ABP06] E. Ackerman, G. Barequet, and R.Y. Pinter. A bijection between permutations and floorplans, and its applications. *Discrete Applied Mathematics*, 154 (12), pp. 1674–1685, 2006.

[AM10] A. Asinowski and T. Mansour. Separable d-permutations and guillotine partitions. *Annals of Combinatorics*, 14, pp. 17–43, 2010.

Alex Pilz

Let S be a set of n points in the plane, labeled from 1 to n. A crossing-free geometric (straight-line) path with vertices on S is *monotonic* if the labels encountered along the path are monotonically increasing. Sakai and Urrutia (*Non-crossing Monotonic Paths in Labeled Point Sets on the Plane*, EuroCG 2015) show that, for any set of n points in general position, there is a monotonic path of length $c(\sqrt{n}-1)$ for $c = 1.0045\ldots$.

We consider a modification of the question in the following way. A crossing-free geometric tree with vertices on S is *monotonic* if the labels encountered during a breath-first traversal of the tree is monotonically increasing or decreasing. Note that the traversal has to be done in the order given by the embedding of the tree (i.e., by the rotation around each vertex). What is the largest k s.t. every point set has a monotonic tree of size at least k. We may also consider depth-first traversal and other variations.

Progress

For monotonic breath-first traversal, we found a family of labelings for points in convex position that arguably allow only trees of size $O(n^{2/3})$. However, we do not have a detailed proof so far. Similarly, we could argue that, for monotonic depth-first traversal, there are labelings of points in convex position that admit only trees of size $O(n^{1/2})$. Again, we do not have worked out all details for proving correctness of the construction.

Csaba Biro

We define two kinds of partial orders between certain closed line segments of $\mathbb{R}^2$. The segments subject to these orders have endpoints $(x_1, 0)$ and (x_2, x_3) such that $x_1 < 0$ and $x_2, x_3 > 0$. First we define the partial order of the first kind. We say, the line segment x (whose endpoints are $(x_1, 0)$ and (x_2, x_3)) is greater than the line segment y (whose endpoints are $(y_1, 0)$ and (y_2, y_3)), if $x_1 < y_1$, $x_2 > y_2$ and every vertical line that intersects both x and y intersects x strictly above y. In particular, if x and y intersect, then they are incomparable.

For the partial order of the second kind, we say, the line segment x is greater than the line segment y, if $x_1 < y_1$, $x_2 < y_2$ and every vertical line that intersects both x and y intersects x strictly above y. In other words, we reverse the ordering rule on the right end of the line segment.

Let P be a poset. If there is a function f such that for $x, y \in P$, $x < y$ if and only is $f(x)$ is less than $f(y)$ in the first kind of line segment ordering, then we say P is a segment order of the first kind. We similarly define segment orders of the second kind.

Question: Is the class of segment orders of the first kind same as the class of segment orders of the second kind?

Alternatively, we can drop the restriction from the definitions that requires that the classes intersect the y axis. The definitions of orders are the identical, except the line segments that are subject to the order have endpoint of the form $(x_1, 0)$, (x_2, x_3) such that $x_1 < x_2$ and $x_3 > 0$. We call these noncentral segment orders of the first kind and of the second kind.

Obviously, every segment order of the first (second) kind is also a noncentral segment order of the first (second) kind. Is it true that either of these class containments is strict?

Progress

Given a partial order of the first kind or second kind and a corresponding representation with segments, we applied transformations to the plane, such as shearings and changing the coordinate system, to obtain different characterizations of these partial orders. Using projective transformations we managed to prove that whenever P is a partial order of the first, respectively second, kind, then the dual P^* of P is a partial order of the first, respectively second, kind, too. Moreover, we explicitly constructed examples of small partial orders that are neither of the first nor second kind. Our main ingredient here is the vertex-edge incidence poset of a graph and a Ramsey argument. The smallest poset obtained this way is the vertex-edge incidence poset of K_9.

Piotr Micek

On-line coloring of unit-length intervals

Presenter presents one unit-length interval on a real line at a time. Algorithm colors it immediately and irrevocably in such a way that intersecting intervals receive distinct colors. Algorithm has a strategy to use at most $2\omega - 1$ colors on any family with clique-size (or chromatic number) ω. (Any greedy strategy will do.) Presenter has a strategy to force $3/2\omega$ colors on a family with clique-size ω. (Easy.) Could we finally improve one of these bounds? I offer 3 beers for a strategy for Algorithm using at most $2\omega - 2$ colors.

Jerry Spinrad

To compute the width of a poset or the independence number of comparability graphs we can use a bipartite matching algorithm which essentially runs in $O(n\,m)$ time.

In some cases where the poset is given with a geometric representation we can do better. As simple examples take interval orders or containment orders of intervals, i.e., 2-dimensional orders, in both cases the complexity drops to $O(n \log n)$.

A general question is to identify more instances where it is helpful for computing the width to have the poset given by a geometric representation. More specifically: Are representations in terms of axis-aligned rectangles or discs helpful?

Piotr Micek

The *Boolean dimension* $\dim_{\mathscr{B}}(P)$ of a poset P was defined by Nešetřil as the minimum positive integer d such that there exist k permutations $L_1, \ldots, L_k$ of the elements of P and a Boolean formula $\phi(z_1, \ldots, z_k)$ so that for every two elements x, y in P, we have

$$x \leq y \text{ in } P \text{ if and only if } \phi([x \leq y \text{ in } L_i]_{i \in [k]}) = 1.$$

It is easy to see that

1. $\dim_{\mathscr{B}}(P) \leq \dim(P)$, for any poset P;
2. $\dim_{\mathscr{B}}(S_n) \leq 4$, where S_n is the standard example of order $n \geq 2$;
3. Boolean dimension of incidence posets is at most 4;
4. Boolean dimension of posets of height 2 is unbounded (by simple counting: there are roughly 2^{n^2} posets on n elements with height 2, while there are only $2^{2^k}(n!)^k$ posets on n elements with Boolean dimension at most k);
5. universal interval orders I_n, i.e. interval orders consisting of all intervals with endpoints within $\{1, \ldots, n\}$, have unbounded interval dimension (the proof is very similar to the proof that they have unbounded usual dimension).

Jarik Nešetřil in [NP89] posed the following problem

Problem: Is it true that planar posets have bounded Boolean dimension?

We were working on this problem before the workshop with Grzegorz Gutowski and Gwenaël Joret. After the workshop we worked together with Heather Smith, Libby Taylor, Tom Trotter and Bartosz Walczak.

We proved that posets with cover graphs of bounded pathwidth have bounded Boolean dimension. This is a little indication for a positive resolution of the problem as the usual dimension is unbounded for this class of posets (cf. Kelly's examples).

[NP89] J. Nešetřil and P. Pudlák, *A note on Boolean dimension of posets*, in Irregularities of partitions, Springer, 1989, pp. 137–140.

Paweł Rzążewski

In the family of representation extension problems we are given a graph G, which belongs to some class of geometric intersection graphs (say, interval graphs) and its partial representation $\mathcal{R}$. In other words, the representation of some vertices is already fixed. We ask for a representation $\mathcal{R}'$ of the whole G, which extends $\mathcal{R}$. Klavík *et al.* [Klav+] showed polynomial-time algorithms deciding the extendability of representations of proper and unit interval graphs.

We study a more general problem. Let us focus on the simplest case of proper interval graphs, but of course we can state analogous questions for other classes.

Problem: A proper interval graph G, a partial representation $\mathcal{R}$ of G, $k \in \mathbb{N}$

Question: Is there a set of at most k intervals in $\mathcal{R}$, whose removal results in an extendible representation of G?

By analyzing minimal obstructions for extendible proper interval representation, we obtained a simple branching algorithm solving the problem in time $3^k \cdot n^{O(1)}$, where n is the number of vertices of G. Can we solve it in polynomial time?

[Kla+] P. Klavík, J. Kratochvíl, Y. Otachi, I. Rutter, T. Saitoh, M. Saumell, and T. Vyskocil, *Extending partial representations of proper and unit interval graphs*, in SWAT 14, LNCS 8503, 253–264, 2014.

Ryan Martin

- Scribe: Piotr Micek

Let Q_n be the Boolean lattice of dimension n. Now $R(Q_m, Q_n)$ is the minimum d such that every red/blue-coloring of elements of Q_d there is a red induced copy of Q_m or blue induced copy of Q_n. Can we improve the bounds from [AW15]?

$$2n \leq R(Q_n, Q_n) \leq n^2 + 2n,$$

$$R(Q_3, Q_3) \in \{7, 8\}.$$

It is known that $R(Q_2, Q_2) = 4$.

[AW15] Maria Axenovich and Stefan Walzer, *Boolean lattices: Ramsey properties and embeddings*, `https://arxiv.org/pdf/1512.05565v1.pdf`.

Tom Trotter

- Scribe: Piotr Micek

A poset Q *arrows* a poset P, if for every two-coloring of a point set of Q, there is a monochromatic induced copy of P in Q. We write $Q \leftarrow P$ to denote that Q arrows P. It is easy to see that every poset P has a poset Q that arrows it. In order to construct Q, simply start with a copy of P and replace each point with a copy of P.

We are interested in three functions capturing the worst scenario posets to arrow:

$$\begin{aligned} f_1(n) &= \max_{\substack{P \\ |P|=n}} \min_{\substack{Q \\ Q \to P}} |Q| \\ f_2(n) &= \max_{\substack{P \\ \text{height}(P)=n}} \min_{\substack{Q \\ Q \to P}} \text{height}(Q) \\ f_3(n) &= \max_{\substack{P \\ \text{width}(P)=n}} \min_{\substack{Q \\ Q \to P}} \text{width}(Q) \end{aligned}$$

Rödl and Nešetřil [NR84] proved that $f_2(n) = 2n - 1$. The lower bound is easy. Indeed, in order to get to a contradiction, suppose that $Q \leftarrow P$ and $\text{height}(Q) \leq 2n - 2$ while $\text{height}(P) = n$. Partition Q into at most $2n - 2$ antichains. Color $n - 1$ antichains red and the other (at most $n - 1$) blue. Clearly, posets induced by each of the colors have height at most $n - 1$ so there cannot be an induced copy of P. For the upper bound they apply the amalgamation technique, see [NR84] for details.

The two other functions were studied by Kierstead and Trotter in [KT87]. For the $f_1(n)$ function the best known bounds are

$$\frac{1}{4} n^2 \leq f_1(n) \leq n^2 - n + 1$$

The upper bound is given basically by the substitution argument described above. The lower bound is witnessed by the poset P consisting of an antichain and chain, both of size $n/2$ with no comparabilities between them.

The most exciting gap stays open for the $f_3(n)$ function:

$$2n \leq f_3(n) \leq n^2$$

The lower bound $2n - 1$ is again easy. But in order to get $2n$ from below one needs to work a bit more, see [KT87].

[KT87] H.A. Kierstead and W.T. Trotter *A Ramsey theoretic problem for finite ordered sets.* Discrete Math. 63 (1987), 217–223.

[NR84] J. Nešetřil and V. Rödl *Combinatorial partitions of finite posets and lattices—Ramsey lattices.* Algebra Universalis, 19 (1984) 106—119.