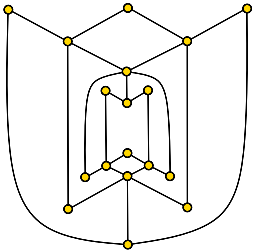
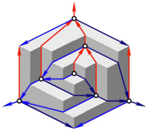
Structural Graph Theory and Dimension





Order and Geometry
Gułtowy 2016

Sparsity Theory for Graphs

Algorithms and Combinatorics 28

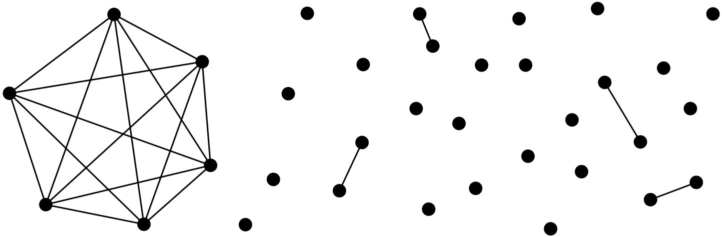
Jaroslav Nešetřil
Patrice Ossona de Mendez

Sparsity

Graphs, Structures, and Algorithms

Springer

When is a graph sparse?



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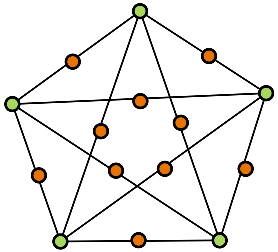
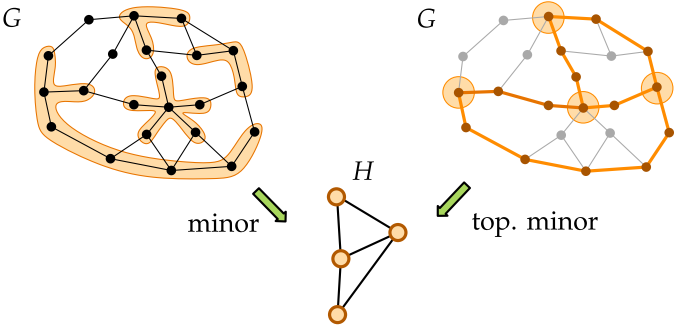


Graph Minors



Theorem [Robertson, Seymour]
Let C be a proper minor-closed class of graphs. Then there exist $H_1, \ldots, H_k$ such that

$$C = \{G : H_i \not\preccurlyeq G \text{ for all } i \in [k]\}.$$

Proper minor-closed graph classes

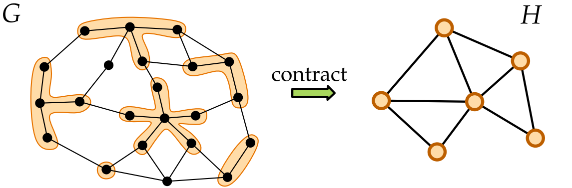
Examples:

- planar graphs
- bounded genus graphs
- graphs of bounded path-width
- graphs of bounded tree-width

Sparse?

- linear many edges
- even their minors have linear many edges!

Shallow Minors



branch sets have radius ≤ 1   1-shallow minor of G

$\mathcal{C}$ class of graphs

$\mathcal{C} \nabla r$ set of r-shallow minors of graphs in $\mathcal{C}$

A class $\mathcal{C}$ has ***bounded expansion*** if there exists a function f s.t. graphs in $\mathcal{C} \nabla r$ have density $\leq f(r)$.

Nowhere dense classes

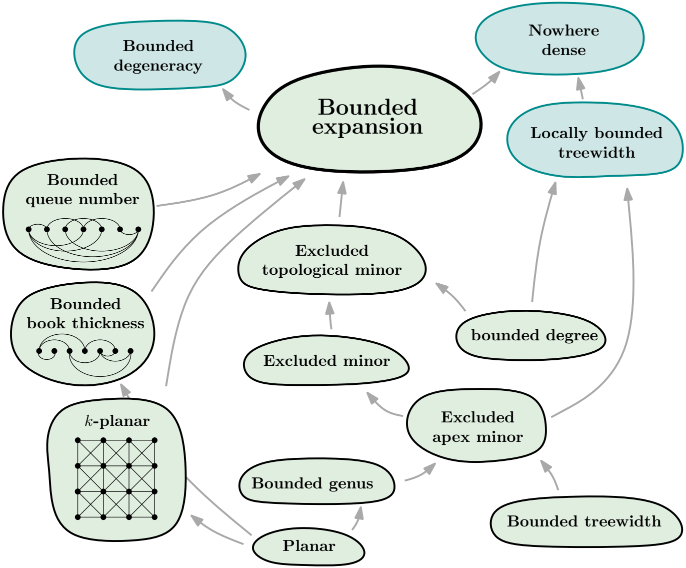
A class $\mathcal{C}$ is *nowhere dense* if for all $r \geq 0$, graphs of $\mathcal{C} \nabla r$ have edge density $\mathcal{O}(n^{\epsilon})$, for each $\epsilon > 0$.

A class $\mathcal{C}$ is *nowhere dense* if for all $r \geq 0$,

$$\mathcal{C} \nabla r \neq \text{ set of all graphs.}$$

A class $\mathcal{C}$ is *somewhere dense* if it is not nowhere dense.

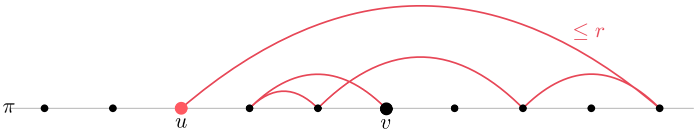
Hierarchy


Bounded degeneracy
Nowhere dense
Bounded expansion
Locally bounded treewidth
Bounded queue number
Excluded topological minor
Bounded book thickness
bounded degree
Excluded minor
k-planar
Excluded apex minor
Bounded genus
Bounded treewidth
Planar

Bounded expansion - characterizations

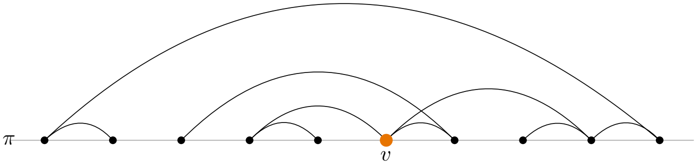
- r-shallow topological minors
- transitive fraternal augmentations
- generalized coloring numbers
- low-treedepth colorings
- neighborhood complexity
- neighborhood covers
- splitter game
- dimension?

Weak Coloring Numbers



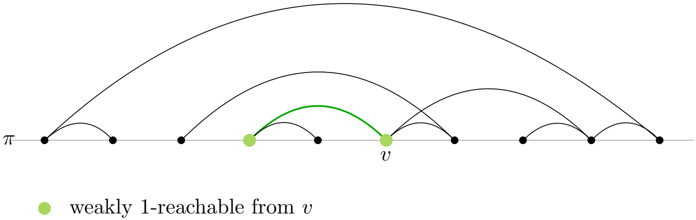
u is weakly r-reachable from v

Weak Coloring Numbers

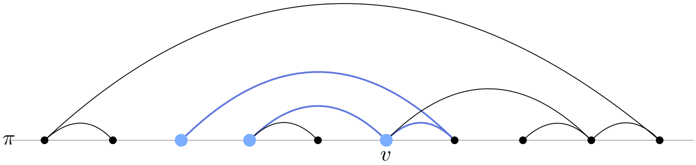


weakly 0-reachable from v

Weak Coloring Numbers

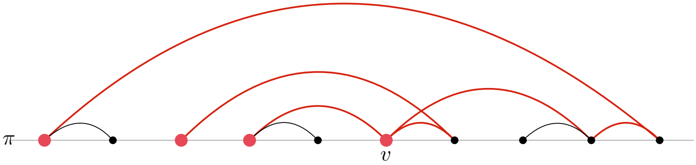


Weak Coloring Numbers



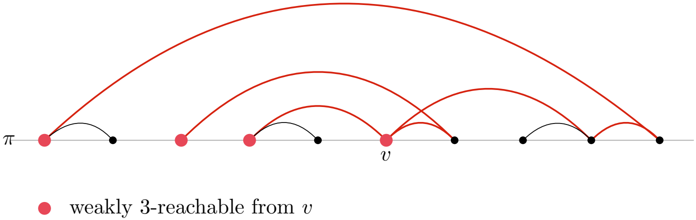
weakly 2-reachable from v

Weak Coloring Numbers



weakly 3-reachable from v

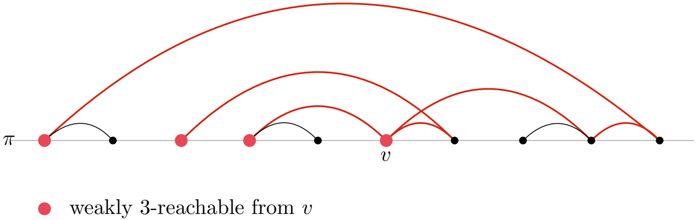
Weak Coloring Numbers



$$\mathrm{wcol}_r(G) := \min_{\pi} \max_{v} |\mathrm{WReach}_r[v, \pi]| .$$

Theorem [Zhu '09]
A class $\mathcal{C}$ has bounded expansion iff there exists a function f such that $\mathrm{wcol}_r(G) \leq f(r)$ for all $r \geq 0$ and $G \in \mathcal{C}$.

Weak Coloring Numbers



$$\mathrm{wcol}_r(G) := \min_{\pi} \max_{v} |\mathrm{WReach}_r[v,\pi]|\,.$$

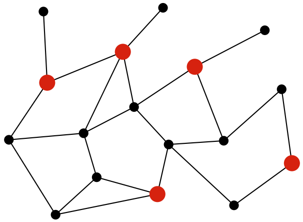
Theorem [Zhu '09]
A class $\mathcal{C}$ is nowhere dense iff for each integer $r \geq 0$ and $\epsilon > 0$, we have $\mathrm{wcol}_r(G) = \mathcal{O}(n^{\epsilon})$ for every $G \in \mathcal{C}$.

Algorithmic Aspects

Dominating Set Problem

Input: Graph G, number k

Problem: Are there k vertices dominating all vertices of G?



NP-complete in general. Is it ***fixed-parameter tractable***? So is there a function f and an algorithm solving the problem in time

$$f(k) \cdot n^{O(1)} \ ?$$

Dominating Set Problem is W[2]-complete

→ unlikely that there exists an FPT for it

NP-complete Graph Problems

- Dominating Set Problem
- k-Colorability
- CLIQUE, INDEPENDET SET
- Steiner tree problem
- k-disjoint paths

General question:
What are the *largest graph classes* on which certain *types of problems* become tractable?

Graph Properties

Goal: Read tractibility of a problem directly off its mathematical description.

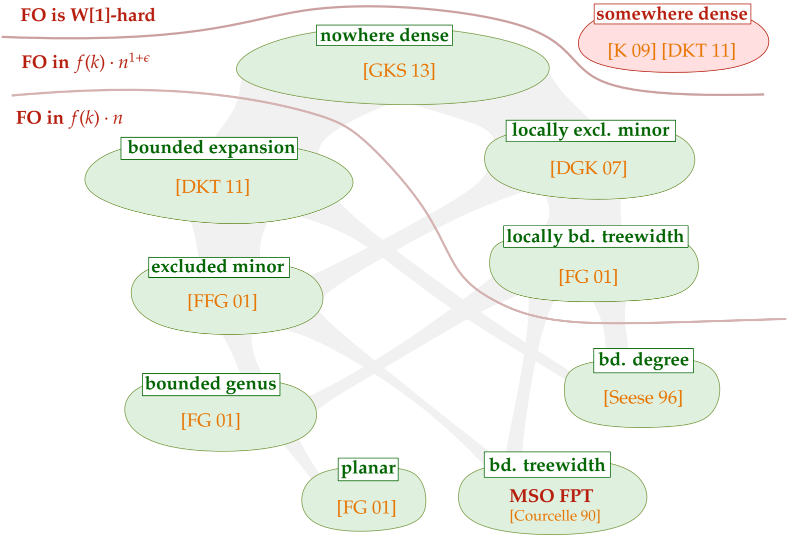
Properties definable in First-Order Logic (FO):

- k-clique, k-independet set
- subgraph containment (for some fixed graph)
- k-dominating set

Properties definable in Monadic Second-Order Logic (MSO):

- connectivity
- hamiltonicity
- k-colorability

Meta-Theorems



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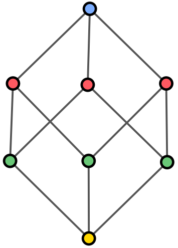
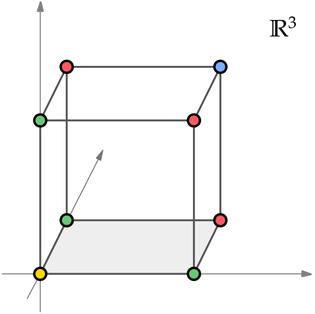
That is to say, the *Dushnik-Miller dimension* and not the *Ore-dimension* (but almost).
In other words, dimension is “Graph Coloring for Grown-ups”.

Dimension

The ***dimension*** of a poset **P** is the least d such that **P** is isomorphic to a subposet of $(\mathbb{R}^d, \leq_d)$.



$\hookrightarrow$



Cover Graphs

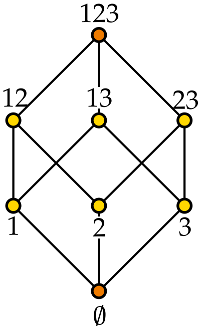
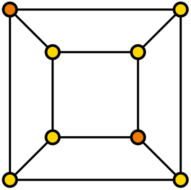
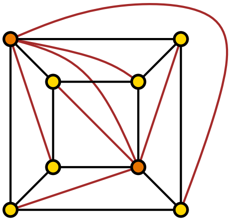


diagram of B_3



cover graph



comparability graph

Width and Dimension

width(**P**) maximum size of an antichain in **P**
height(**P**) maximum size of a chain in **P**

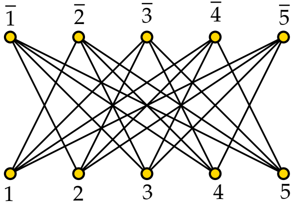
Theorem [Dilworth '50]

$$\dim(\mathbf{P}) \leq \text{width}(\mathbf{P}).$$

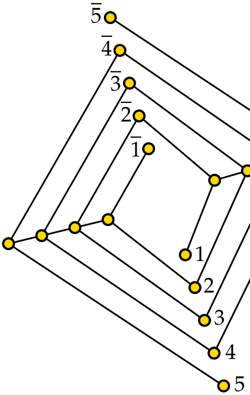
"Large-dimensional posets are *wide*"

but not necessarily *tall*:

S_5



Kelly's examples



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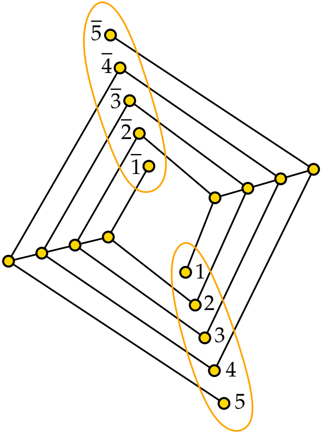

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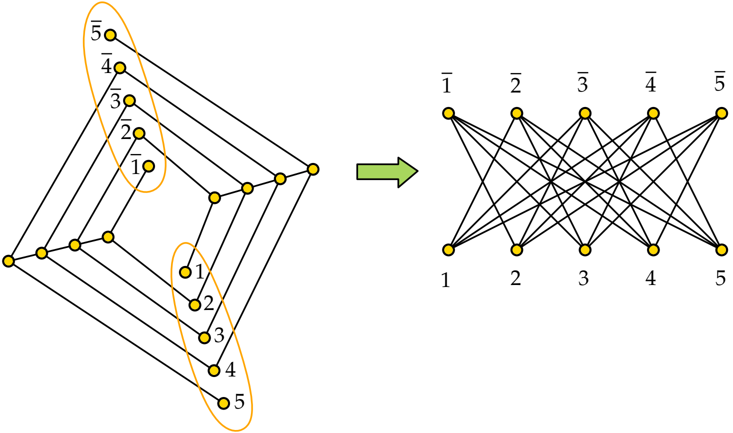

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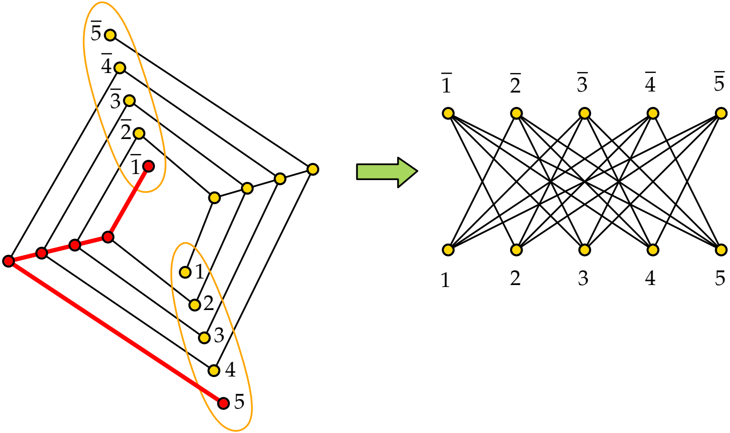


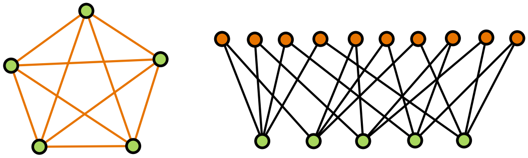
General Question

"Do large-dimensional posets with *sparse* cover graphs have to be *tall*?"

Answer: Yes and No

Theorem [Streib, Trotter, 2014]
The dimension of posets with planar cover graphs is bounded in their height.

Incidence Posets of graphs:



K_5 $\mathbf{P}_{K_5}$

Theorem [Dushnik, Miller, '41]

$$\dim(\mathbf{P}_{K_n}) \geq \log\log n.$$

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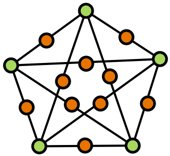
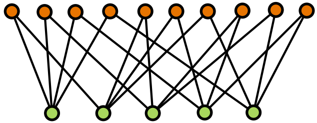


cover($\mathbf{P}_{K_5}$)



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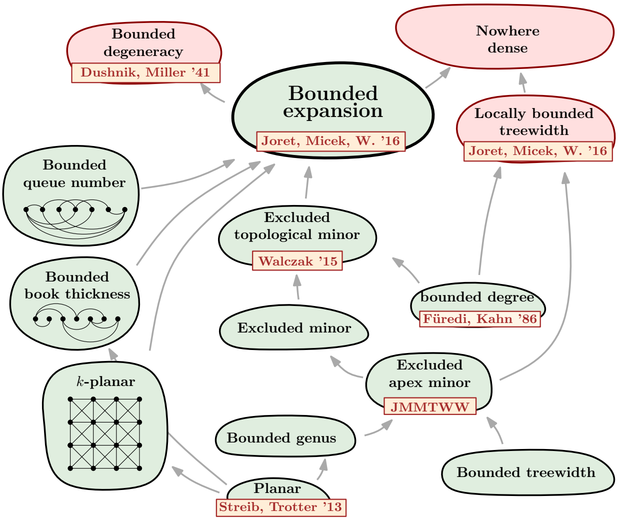
Cover Graphs and Dimension



Weak Coloring Numbers and Dimension

Theorem [Joret, Micek, W., 2016+]
Let $\mathbf{P}$ be a poset of height at most h with a cover graph G such that $\mathrm{wcol}_{3h}(G) \leq c$. Then

$$\dim(\mathbf{P}) \leq 6^c.$$

Graph property	$\mathrm{wcol}_r(G)$	
bounded genus	$\mathcal{O}(r^3)$	**[vH-OdM-Qu-R-S, 2016+]**
treewidth t	$\mathcal{O}(r^t)$	**[GKRSS, 2016]**
no K_n minor	$\mathcal{O}(r^{n-1})$	**[vH-OdM-Qu-R-S, 2016+]**
no K_n top. minor	$2^{\mathcal{O}(r \log r)}$	**[KPRS, 2016]**
bd. expansion	$f(r)$	**[Zhu, 2009]**

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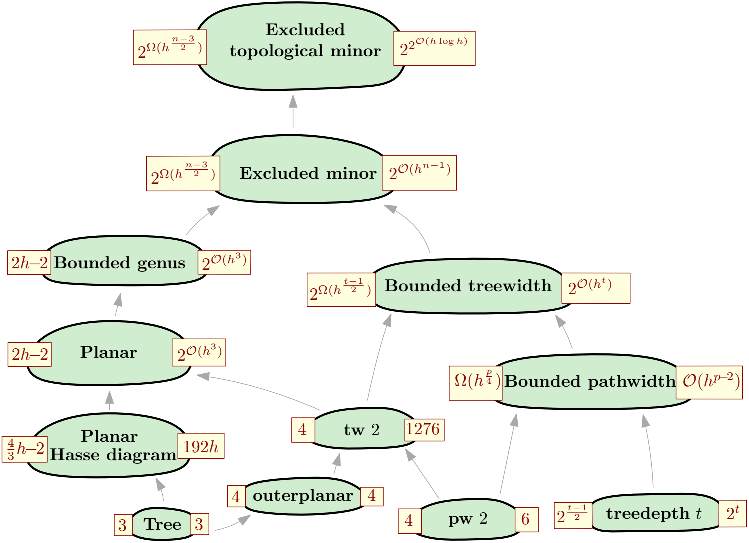
bd. expansion	$f(r)$	$g(h)$

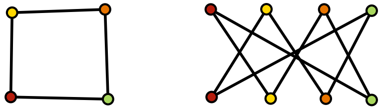
Current best bounds



Nowhere dense cover graphs

Theorem [Joret, Micek, W., 2016]
There are height-2 posets with cover graphs in a **nowhere dense** class $\mathcal{C}$ such that their dimension is unbounded.

Adjacency posets:



G $\qquad$ $\mathbf{AP}_G$

Lemma: $\chi(G) \leq \dim(\mathbf{AP}_G)$.

$\mathcal{C}$ = {graphs G with $\Delta(G) \leq \text{girth}(G)$}.

- nowhere dense , unbounded χ
- $\implies$ $\dim(\mathbf{AP}_G)$ is unbounded for $G \in \mathcal{C}$
- $G \in \mathcal{C} \implies$ cover graph of $\mathbf{AP}_G$ in $\mathcal{C}$

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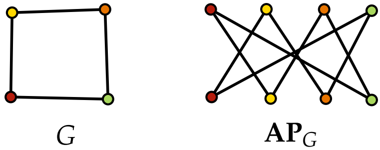

Lemma: $\chi(G) \leq \dim(\mathbf{AP}_G)$.

$\mathcal{C}$ = {graphs G with $\Delta(G) \leq \text{girth}(G)$}.

- has **locally bounded treewidth**, unbounded χ
- $\implies \dim(\mathbf{AP}_G)$ is unbounded for $G \in \mathcal{C}$
- $G \in \mathcal{C} \implies$ cover graph of $\mathbf{AP}_G$ in $\mathcal{C}$

Open Problems

Conjecture
A monotone class C has bounded expansion iff
for each $h \geq 1$, posets of height at most h with cover graphs in C have bounded dimension.

Problem
Let $\mathcal{P}$ be a class of height-2 posets with unbounded average degree. Is the dimension of subposets of posets in $\mathcal{P}$ necessarily unbounded?

Open Problems

Conjecture
Posets $\mathbf{P}$ of bounded height with cover graphs in a nowhere dense class have dimension

$$\dim(\mathbf{P}) \leq \mathcal{O}(n^{\epsilon}),$$

for each $\epsilon > 0$.

Fact:
For every monotone somewhere dense class C, there exists h such that there are posets $\mathbf{P}$ of height at most h and

$$\dim(\mathbf{P}) = \Omega(n^{1/2}).$$

Open Problems

Problem
Large-dimensional posets with sparse cover graphs have to be *tall*.
What else?

Theorem [Howard, Streib, Trotter, Walczak, Wang, 2016+]
Large-dimensional posets with planar cover graphs have to contain a large $\mathbf{k} + \mathbf{k}$.

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Thank You